The Harmonic Balance Method in ***Manlab-4***

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Summary of the method :

- Find the periodic solution of a differential system

$$\mathbf{f}(t, \mathbf{z}, \dot{\mathbf{z}}, \lambda) = 0 \qquad \mathbf{z}(t) \in \mathbb{R}^n$$

by using a high order Fourier series expansion

$$\mathbf{z}(t) = Z_0 + \sum_{k=1}^{H} Z_{c,k} \cos(k\omega t) + \sum_{k=1}^{H} Z_{s,k} \sin(k\omega t)$$

- The resulting (nonlinear) algebraic system on the Fourier coefficients reads :

$$\mathbf{R}(\mathbf{U}) = 0 \qquad \text{with} \qquad \mathbf{U} = [Z_0, Z_{c,k}, Z_{s,k}, \omega, \lambda] = [\mathbf{Z}, \omega, \lambda]$$

- Then the solution is continued with respect to λ using the Asymptotic Numerical Method (ANM).

The cornerstone

- Harmonic Balance Method HBM : insert
  $\theta(t) = \theta_0 + \sum_{k=1}^{H} \theta_{c,k} \cos(k\omega t) + \sum_{k=1}^{H} \theta_{s,k} \sin(k\omega t)$
  into, for example,

$$R(\theta(t), \lambda) := \ddot{\theta} + \lambda\dot{\theta} + \theta^3 + \sin(\theta) = 0$$

  and "balance" the harmonics !

- Asymptotic Numerical Method ANM : insert
  $u(a) = u_0 + a\, u_1 + a^2\, u_2 + \cdots + a^N\, u_N$
  into

$$R(u, \lambda) := u + u^2 + \frac{tan(u)}{1+u} - \lambda = 0$$

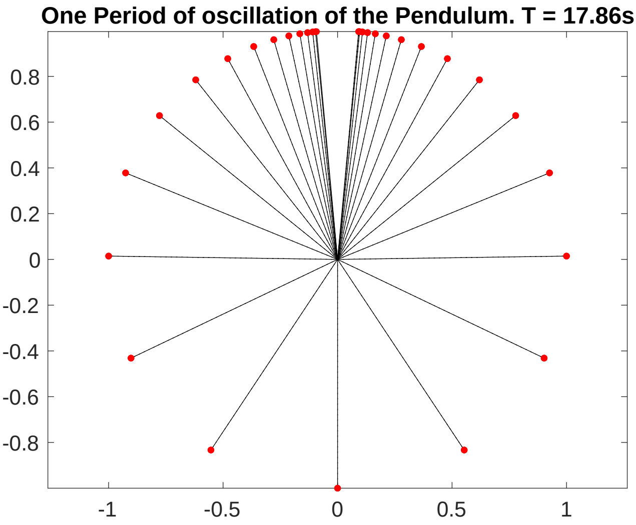
  and collect term with the same powers !

How to :

- use Automatic Differentiation to do the job : nice for the user but poor efficiency.
- do a quadratic recast of the equation : then the job becomes easy and efficient for ANM and HBM !

Live demonstration.

Pendulum

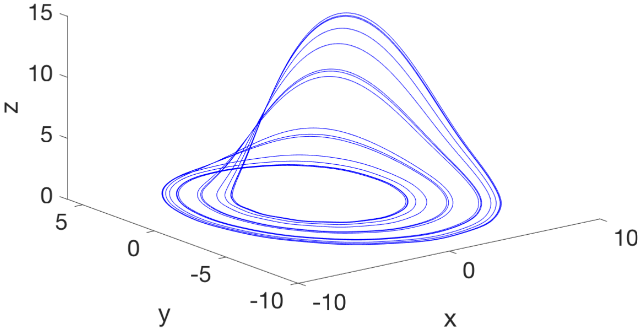

One Period of oscillation of the Pendulum. T = 17.86s
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-0.5
0
0.5
1

Rössler system

A kind of Lorenz system. When $b = 0.2$, a cascade of period doubling bifurcations occurs leading to a chaotic behavior.

$$\begin{cases} \dot{x} = -y - z \\ \dot{y} = x + by \\ \dot{z} = b + z(x - a) \end{cases}$$

On the right : Phase diagram after 4 period doublings.

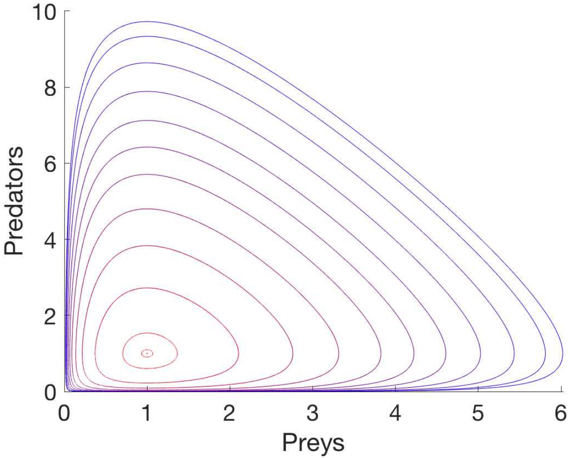


Predator-Prey model

A predator-prey model. The periodic trajectories "turn" around the center $(x, y) = (1, 1)$.

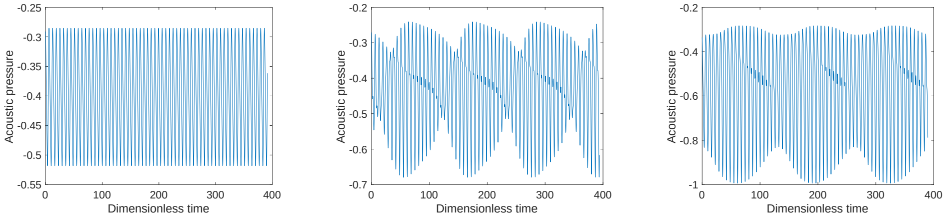
$$\begin{cases} \dot{x} = \dfrac{1}{2}x(1-y) \\ \dot{y} = y(x-1) \end{cases}$$

On the right : Several trajectories.



Saxophone model : multiphonics

After listening to some sounds, these are some time-domain signals of the internal acoustic pressure.



Overview of ***Manlab-4***

Equations (E)

Quadratic recast of the equations (E_q)

If X(t) is periodic, quasi-periodic, or discretized on time samples

(E_q) is projected
on a suitable basis of representation:
by Harmonic Balance Method (HBM),
collocation, etc

A quadratic algebraic system ($E_{q,alg}$) is obtained.

Unknowns (X)

Type of the unknowns:
Algebraic, Function of the time

The continuation of this quadratic algebraic system ($E_{q,alg}$) is performed using the Asymptotic Numerical Method.

Quadratic recast in ODE for HBM

Framework for Ordinary Differential Equations (ODE) :

- To find the periodic solution of the ODE system

$$0 = f(\mathbf{z}, \lambda) - \dot{\mathbf{z}} \qquad \mathbf{z} \in \mathbb{R}^n$$

- first introduce auxiliary variables $\mathbf{z}_\mathrm{f} = [\mathbf{z}; \mathbf{z}_\mathrm{a}]$ $\rightarrow$ quadratic DAE system

$$0 = c_0 + \lambda c_1 + \lambda^2 c_2 + l_0(\mathbf{z}_\mathrm{f}) + \lambda l_1(\mathbf{z}_\mathrm{f}) + q(\mathbf{z}_\mathrm{f}, \mathbf{z}_\mathrm{f}) - m(\dot{\mathbf{z}}_\mathrm{f})$$

- then apply Fourier series expansion (HBM)

$$\mathbf{z}_\mathrm{f}(t) = Zf_0 + \sum_{k=1}^{H} Zf_{c,k} \cos(k\omega t) + \sum_{k=1}^{H} Zf_{s,k} \sin(k\omega t)$$

- the vector of all Fourier coefficients is written

$$\mathbf{Z}_\mathrm{f} = [Zf_0, Zf_{c,k}, Zf_{s,k}]$$

- The resulting algebraic system on the Fourier coefficients is of the form

$$\mathbf{R}(\mathbf{Z}_\mathrm{f},\omega,\lambda)=\mathbf{C}_0+\lambda\mathbf{C}_1+\lambda^2\mathbf{C}_2+\mathbf{L}_0(\mathbf{Z}_\mathrm{f})+\lambda\mathbf{L}_1(\mathbf{Z}_\mathrm{f})+\mathbf{Q}(\mathbf{Z}_\mathrm{f},\mathbf{Z}_\mathrm{f})-\omega\mathbf{M}(\mathbf{Z}_\mathrm{f})$$

- This system is already quadratic ! in all the unknowns $\mathbf{Z}_\mathrm{f},\omega$ and λ.
- HBM gives as many equations as Fourier coefficients in $\mathbf{Z}_\mathrm{f}$. The additional unknown ω is compensated by a phase equation : $\dot{\mathbf{z}}_i(0)=0$.

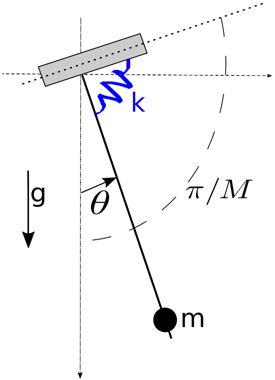
- Advantages of ODE framework : stability available.
- Drawback : double size system for mechanical systems with second order time derivatives

The examples of the pendulum

Dimensionless parameters : $m = 1$ and $g = 1$.

Energy of the free system :
$H(\theta, \dot{\theta}) = \frac{\dot{\theta}^2}{2} + 1 - \cos(\theta) + \frac{k}{2}(\theta - \frac{\pi}{M})^2$

Equation of the motion :
$\ddot{\theta} + \xi\dot{\theta} + \sin(\theta) + k(\theta - \frac{\pi}{M}) = F\cos(\omega t)$



Steps :

- M, k and F are constant, development about $\theta = 0$:
  $\ddot{\theta} + \xi\dot{\theta} + \theta - \frac{\theta^3}{6} + k(\theta - \frac{\pi}{M}) = F\cos(\omega t)$
- M, k and ω are constant, development about $\theta = 0$:
  $\ddot{\theta} + \xi\dot{\theta} + \theta - \frac{\theta^3}{6} + k(\theta - \frac{\pi}{M}) = F\cos(\omega t)$
- M and k are constant, free system without simplification :
  $\ddot{\theta} + \sin(\theta) + k(\theta - \frac{\pi}{M}) = 0$

Basic example : Simplified Pendulum

Forced pendulum subject to an angular spring, developed around $\theta = 0$:

$$\ddot{\theta} + \xi\dot{\theta} + (\theta - \frac{\theta^3}{6}) + k(\theta - \frac{\pi}{M}) = F\cos(\lambda t)$$

First order ODE :

$$\begin{aligned} \dot{\theta} &= \phi \\ \dot{\phi} &= -\xi\phi - (\theta - \tfrac{\theta^3}{6}) - k(\theta - \tfrac{\pi}{M}) + F\cos(\lambda t) \end{aligned}$$

The auxiliary variable $\psi = \theta^2$ is added.
It yields the full quadratic DAE to give to ***Manlab-4*** :

$$\begin{array}{lll} 0 = & \phi & -\dot{\theta} \\ 0 = & -\xi\phi - (\theta - \frac{\psi\theta}{6}) - k(\theta - \frac{\pi}{M}) + F\cos(\lambda t) & -\dot{\phi} \\ 0 = & \psi - \theta^2 & \end{array}$$

General framework

- Find the periodic solution of Implicit Differential Algebraic Equations (IDAE) :

$$g(t, \mathbf{z}, \dot{\mathbf{z}}, \ddot{\mathbf{z}}, \lambda) = 0 \qquad \mathbf{z}(t) \in \mathbb{R}^n$$

- first, introduce auxiliary variables $\mathbf{z}_\mathrm{f} = [\mathbf{z}; \mathbf{z}_\mathrm{a}]$ $\quad \rightarrow$ quadratic DAE system

$$c_0 + \lambda c_1 + \lambda^2 c_2 + l_0(\mathbf{z}_\mathrm{f}) + \lambda l_1(\mathbf{z}_\mathrm{f}) + q(\mathbf{z}_\mathrm{f}, \mathbf{z}_\mathrm{f}) + d(\dot{\mathbf{z}}_\mathrm{f}) + \lambda d_1(\dot{\mathbf{z}}_\mathrm{f}) + dd(\ddot{\mathbf{z}}_\mathrm{f}) = 0$$

- then apply HBM
- The resulting algebraic system on the Fourier coefficients is of the form

$$\begin{aligned}\mathbf{R}(\mathbf{Z}_\mathrm{f}, \omega, \lambda) = {} & \mathbf{C}_0 + \lambda \mathbf{C}_1 + \lambda^2 \mathbf{C}_2 + \mathbf{L}_0(\mathbf{Z}_\mathrm{f}) + \lambda \mathbf{L}_1(\mathbf{Z}_\mathrm{f}) + \mathbf{Q}(\mathbf{Z}_\mathrm{f}, \mathbf{Z}_\mathrm{f}) \\ & + \omega \mathbf{D}(\mathbf{Z}_\mathrm{f}) + \lambda\omega \mathbf{D}_1(\mathbf{Z}_\mathrm{f}) + \omega^2 \mathbf{DD}(\mathbf{Z}_\mathrm{f})\end{aligned}$$

- the auxiliary variables $\Lambda = \lambda\omega$ and $\Omega = \omega^2$ are added to make the algebraic system quadratic.

Simple example : Simplified Pendulum

Forced pendulum subject to an angular spring, developed around $\theta = 0$:

$$\ddot{\theta} + \xi\dot{\theta} + (\theta - \frac{\theta^3}{6}) + k(\theta - \frac{\pi}{M}) = F\cos(\lambda t)$$

The auxiliary variable $\psi = \theta^2$ is added.
It yields the full quadratic DAE to give to ***Manlab-4*** :

$$\begin{aligned} 0 &= \ddot{\theta} + \xi\dot{\theta} + (\theta - \tfrac{\theta\psi}{6}) + k(\theta - \tfrac{\pi}{M}) - F\cos(\lambda t) \\ 0 &= \psi - \theta^2 \end{aligned}$$

Conservative systems

- equation of a conservative system

$$\mathbf{M}\ddot{\mathbf{z}} + f_{nl}(\mathbf{z}) = 0$$

  It already defines a family of periodic solutions without the need of any continuation parameter !

- add artificial damping (unfolding terms) to regularize the continuation

$$\mathbf{M}\ddot{\mathbf{z}} + \lambda\dot{\mathbf{z}} + f_{nl}(\mathbf{z}) = 0$$

  Once solved, λ is found to be zero since no periodic solution exist otherwise.

Manlab-4.0 can take into account these two terms $\mathbf{M}\ddot{\mathbf{z}}$ and $\lambda\dot{\mathbf{z}}$ automatically.

However, the stability analysis of these systems is no more available.

A free Pendulum

Free pendulum subject to an angular spring :

$$\ddot{\theta} + \sin(\theta) + k(\theta - \frac{\pi}{M}) = 0$$

Add the unfolding term $\lambda\dot{\theta}$ to regularize the continuation :

$$\ddot{\theta} + \lambda\dot{\theta} + \sin(\theta) + k(\theta - \frac{\pi}{M}) = 0$$

The auxiliary variables $s = \sin(\theta)$ and $c = \cos(\theta)$ are added.
It yields the "quadratic" recast with differential form (when needed) :

$$\begin{aligned} 0 &= \ddot{\theta} + \lambda\dot{\theta} + s + k(\theta - \tfrac{\pi}{M}) & & \\ 0 &= s - \sin(\theta) & \qquad 0 &= \mathrm{d}s - c\mathrm{d}\theta \\ 0 &= c - \cos(\theta) & \qquad 0 &= \mathrm{d}c + s\mathrm{d}\theta \end{aligned}$$

Initialization

- The initialization on a periodic solution branch is generally the most difficult part in ***Manlab-4***.
- Several remarks :
  - Forced system are usually easy to initialize. (far from the resonances)
  - If an analytic approximation with $H = 1$ is available, it usually works.
  - For autonomous systems, starting at low amplitude with a linear guess usually works.

Vector of unknown and implementation in ***Manlab-4***

- In ***Manlab-4***, the full vector of unknowns is $\mathbf{U}_\mathrm{f} = [\mathbf{Z}; \omega; \lambda; \mathbf{Z}_\mathrm{a}; \Omega; \Lambda]$ where $\mathbf{Z}$ and $\mathbf{Z}_\mathrm{a}$ are the Fourier coefficients of the main and auxiliary variables.
- The matrices of Fourier coefficients $\mathbf{Z}$ and $\mathbf{Z}_\mathrm{a}$ are given in a unique matrix $\mathbf{Z}_\mathrm{f}$ that contains in column the Fourier developments of all the variables :

$$\mathbf{Z} = \begin{bmatrix} Z_0 & \text{constant} \\ Z_{c,1} & \text{first cosine} \\ Z_{c,2} & \text{second cosine} \\ \vdots & \vdots \\ Z_{c,H} & \text{last cosine} \\ Z_{s,1} & \text{first sine} \\ \vdots & \vdots \\ Z_{s,H} & \text{last sine} \end{bmatrix}
\qquad
\mathbf{Z}_\mathrm{f} = \begin{array}{c} \\ \\ \end{array}\left[\begin{array}{ccccccc}
\overbrace{}^{\mathbf{z}^{(1)}} & \overbrace{}^{\mathbf{z}^{(2)}} & \overbrace{}^{\cdots} & \overbrace{}^{\mathbf{z}^{(\mathrm{nz})}} & \overbrace{}^{\mathbf{z}_\mathrm{a}^{(1)}} & \overbrace{}^{\cdots} & \overbrace{}^{\mathbf{z}_\mathrm{a}^{(\mathrm{nza})}} \\
Z_0^{(1)} & Z_0^{(2)} & & Z_0^{(\mathrm{nz})} & Za_0^{(1)} & & Za_0^{(\mathrm{nza})} \\
Z_{c,1}^{(1)} & Z_{c,1}^{(2)} & & Z_{c,1}^{(\mathrm{nz})} & Za_{c,1}^{(1)} & & Za_{c,1}^{(\mathrm{nza})} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
Z_{c,H}^{(1)} & Z_{c,H}^{(2)} & & Z_{c,H}^{(\mathrm{nz})} & Za_{c,H}^{(1)} & & Za_{c,H}^{(\mathrm{nza})} \\
Z_{s,1}^{(1)} & Z_{s,1}^{(2)} & & Z_{s,1}^{(\mathrm{nz})} & Za_{s,1}^{(1)} & & Za_{s,1}^{(\mathrm{nza})} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
Z_{s,H}^{(1)} & Z_{s,H}^{(2)} & & Z_{s,H}^{(\mathrm{nz})} & Za_{s,H}^{(1)} & & Za_{s,H}^{(\mathrm{nza})}
\end{array}\right]$$

Useful functions built-in ***Manlab-4***

- `init_U0(`$\mathbf{Z}_\mathrm{f}, \omega, \lambda$`)` : gives the vector $\mathbf{U}_\mathrm{f}$ of all the unknowns.
- `init_Hdiff(`$\mathbf{U}_\mathrm{f}$`)` : gives the vector $\mathbf{U}_\mathrm{f}$ of all the unknowns for a different harmonics number.
- `get_Ztot(`$\mathbf{U}_\mathrm{f}$`)` : gives the vector $\mathbf{Z}_\mathrm{f}$,the frequency ω and the continuation parameter λ (inverse of `init_U0`).
- `getcoord('cos',i,h)` : gives the coordinate of $Z_{c,h}^{(i)}$ in the total vector of unknowns $\mathbf{U}_\mathrm{f}$.

End of the user part.

Hill's Method for a frequency domain stability algorithm

Let $\dot{\mathbf{z}} = \mathbf{F}(\mathbf{z})$ be an Ordinary Differential Equation and let $t \mapsto \mathbf{z}_0(t)$ be a periodic solution of this system.

- Perturbation :

$$\mathbf{z}(t) = \mathbf{z}_0(t) + \varepsilon(t)$$

- Linear development of F around $\mathbf{z}_0(t)$:

$$\dot{\varepsilon}(t) = \mathrm{dF}_{\mathbf{z}_0}(t)\varepsilon(t)$$

- Floquet theorem : $\varepsilon(t) = \mathrm{e}^{\alpha t}\mathbf{p}(t)$, with $\mathbf{p}$ a periodic function. It gives :

$$\alpha\mathbf{p}(t) + \dot{\mathbf{p}}(t) = \mathrm{dF}_{\mathbf{z}_0}(t)\mathbf{p}(t)$$

- Hill's method : $\mathbf{p}$ and $\mathrm{dF}_{\mathbf{z}_0}(t)$ are periodic. Let $\mathbf{P}$ be the infinite Fourier development of $\mathbf{p}(t)$.

$$\mathbf{HP} = \alpha\mathbf{P}$$

Eigenvalue problem with Hill's matrix **H**.
***H** is the Jacobian matrix of the algebraic system obtained after the Harmonic Balance.*

The SystODE class of ***Manlab-4*.1.3.**

Framework for Ordinary Differential Equations (ODE) :

- Find the solutions of the ODE system

$$0 = \mathbf{f}(\mathbf{z}, \lambda) - \dot{\mathbf{z}} \qquad\qquad \mathbf{z} \in \mathbb{R}^n$$

- first, recast quadratic $\mathbf{z}_\mathrm{f} = [\mathbf{z}; \mathbf{z}_\mathrm{a}]$ $\quad\rightarrow$ quadratic DAE system

$$0 = c_0 + \lambda c_1 + \lambda^2 c_2 + l_0(\mathbf{z}_\mathrm{f}) + \lambda l_1(\mathbf{z}_\mathrm{f}) + q(\mathbf{z}_\mathrm{f}, \mathbf{z}_\mathrm{f}) - m(\dot{\mathbf{z}}_\mathrm{f})$$

- Equilibrium are given by the solutions to the equations :

$$0 = c_0 + \lambda c_1 + \lambda^2 c_2 + l_0(\mathbf{z}_\mathrm{f}) + \lambda l_1(\mathbf{z}_\mathrm{f}) + q(\mathbf{z}_\mathrm{f}, \mathbf{z}_\mathrm{f})$$

- Periodic solutions are given by the solutions to the equations :

$$\mathbf{R}(\mathbf{Z}_\mathrm{f}, \omega, \lambda) = \mathbf{C}_0 + \lambda \mathbf{C}_1 + \lambda^2 \mathbf{C}_2 + \mathbf{L}_0(\mathbf{Z}_\mathrm{f}) + \lambda \mathbf{L}_1(\mathbf{Z}_\mathrm{f}) + \mathbf{Q}(\mathbf{Z}_\mathrm{f}, \mathbf{Z}_\mathrm{f}) - \omega \mathbf{M}(\mathbf{Z}_\mathrm{f})$$

After condensation, stability information are directly available :

- For equilibrium through the eigenvalues of the Jacobian matrix $\mathbf{J}$.
  - At a Hopf point $\mathbf{z}_0$ there exists $\mathbf{P} \neq 0$ such that $\mathbf{JP} = i\omega\mathbf{P}$.
  - It gives the starting periodic orbit after Hopf bifurcation of the form $\mathbf{z}(t) = \mathbf{z}_0 + \mathbf{P}\exp(i\omega t)$.
- For periodic solution through the eigenvalues of Hill's matrix $\mathbf{H}$.
  - At a Neimark-Sacker (NS) point $\mathbf{Z}_0$ there exists $\mathbf{P} \neq 0$ such that $\mathbf{HP} = i\omega_2\mathbf{P}$.
  - It gives the starting quasi-periodic orbit after NS bifurcation of the form $\mathbf{Z}(t) = \mathbf{Z}_0 + \mathbf{P}\exp(i\omega_2 t)$.

These initializations are automatized in ***Manlab-4*.1.3.**