The Asymptotic Numerical Method in *Manlab-4*

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Aim of the talk :

Describe the so-called **Asymptotic Numerical Method**,

a continuation method
using high order Taylor series expansions.

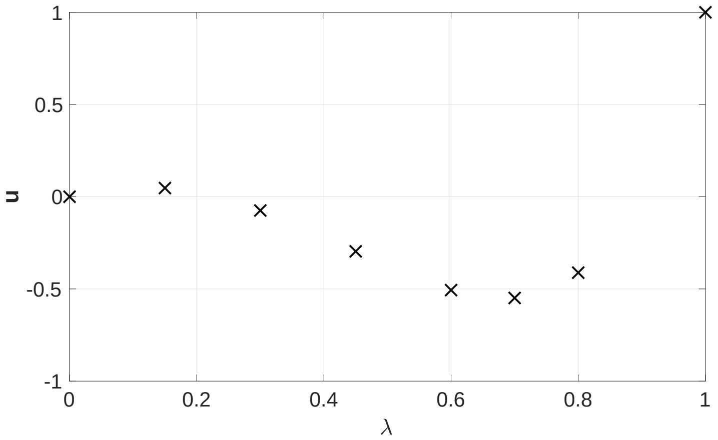
Continuation

Goal : determine solution branches of

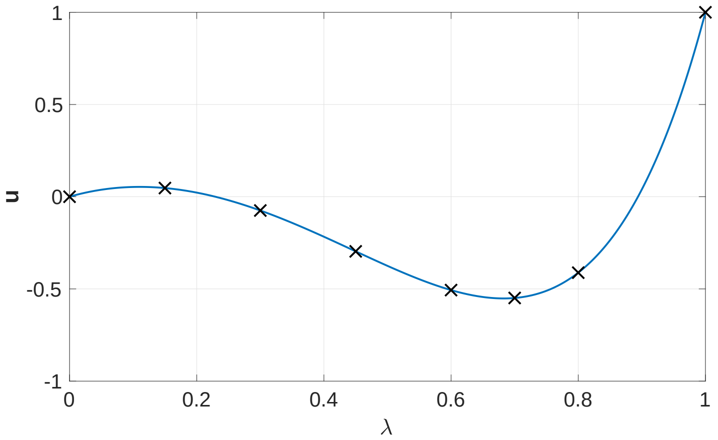
$$\mathbf{R}(\mathbf{u}, \lambda) = 0$$

with $\mathbf{u} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$
and $\mathbf{R} : \mathbb{R}^{n+1} \mapsto \mathbb{R}^n$ is a smooth function.

Continuation



Continuation



The cornerstone

To compute the Taylor series of the solution-branch :

- Insert the Taylor series

$$u(a) = u_0 + a\, u_1 + a^2 u_2 \; + \cdots + a^N\, u_N$$

- Into the algebraic equation

$$R(u, \lambda) := u + u^2 + \frac{tan(u)}{1+u} - \lambda = 0$$

- And collect terms with the same powers !

Live demonstration.

Logistic Map

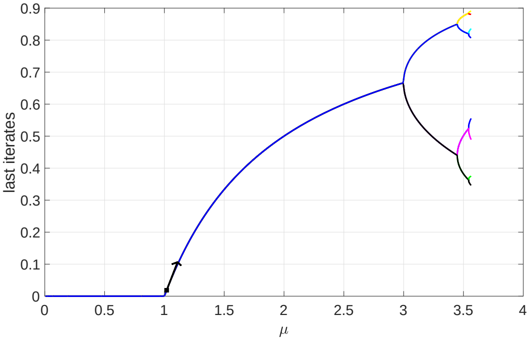
Compute the iterates of the logistic map

$$f(x) = \mu x(1 - x)$$

The system solved is

$$\begin{cases} x_1 = f(x_0) \\ x_2 = f(x_1) \\ \vdots \\ x_{N-1} = f(x_{N-2}) \\ x_N = f(x_{N-1}) \end{cases}$$

with $x_0 = 0.5$ and $N = 5000$.



Pendulum

Compute the orbit of the pendulum

$$\left\{ \begin{array}{ll} \dot{\theta} & = \phi \\ \dot{\phi} & = -\sin(\theta) \end{array} \right.$$

with $\theta(0) = \frac{\pi}{2}$ and $\phi(0) = 0$. The system is solved with a λ-scheme :

$$\left\{ \begin{array}{ll} \theta_{n+1} & = \theta_n + h\phi_{n+\lambda} \\ \phi_{n+1} & = \phi_n - h\sin(\theta_{n+\lambda}) \end{array} \right.$$

with $x_{n+\lambda} = (1-\lambda)x_n + \lambda x_{n+1}$, and $h = \frac{2\pi}{100}$.

Pendulum

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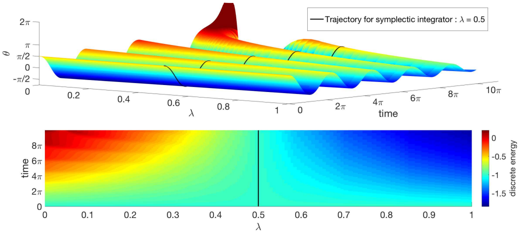
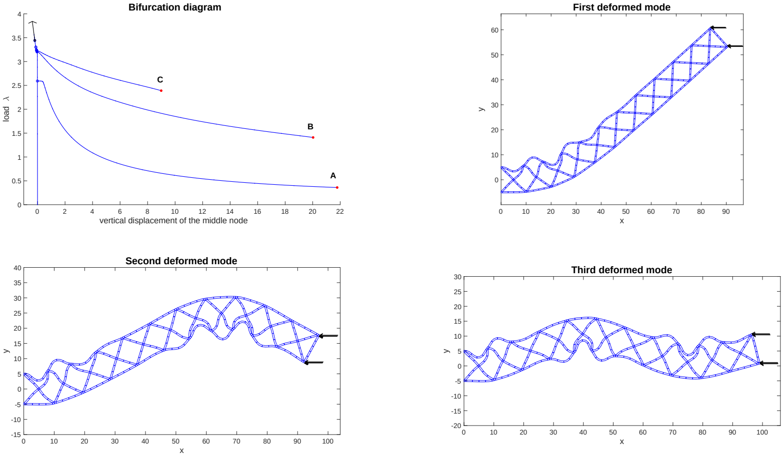


Deformation of a complex structure



Overview of ***Manlab-4***

Equations (E)

Quadratic recast of the equations (E_q)

If X(t) is periodic, quasi-periodic, or discretized on time samples

(E_q) is projected
on a suitable basis of representation:
by Harmonic Balance Method (HBM),
collocation, etc

A quadratic algebraic system ($E_{q,alg}$) is obtained.

Unknowns (X)

Type of the unknowns:
Algebraic, Function of the time

The continuation of this quadratic algebraic system ($E_{q,alg}$) is performed using the Asymptotic Numerical Method.

Continuation

Goal : determine solution branches of

$$\mathbf{R}(\mathbf{U}) = \mathbf{R}(\mathbf{u}, \lambda) = 0$$

where $\mathbf{U} = [\mathbf{u}, \lambda]$.

with $\mathbf{u} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$
and $\mathbf{R} : \mathbb{R}^{n+1} \mapsto \mathbb{R}^n$ is a smooth function.

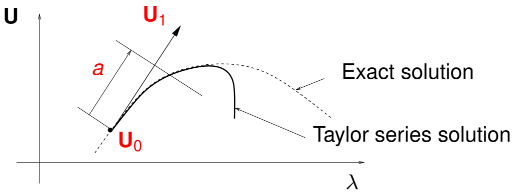
Taylor series based continuation

- let $\mathbf{U}_0$ be a regular point with $||\mathbf{R}(\mathbf{U}_0)|| < \varepsilon_R$ (tolerance).
- let $\mathbf{U}_1$ be the tangent at $\mathbf{U}_0$.
- let a be the pseudo arc length parameter $a = \mathbf{U}_1^T.(\mathbf{U} - \mathbf{U}_0)$.

Implicit Function theorem : The solution branch passing through $\mathbf{U}_0$ may be represented as a (truncated) Taylor series with respect to the pseudo-arclength parameter a.

$$\mathbf{U}(a) = \mathbf{U}_0 + a\mathbf{U}_1 + a^2\mathbf{U}_2 + \cdots + a^N\mathbf{U}_N \qquad \text{and } N = 20 \text{ or } 30$$

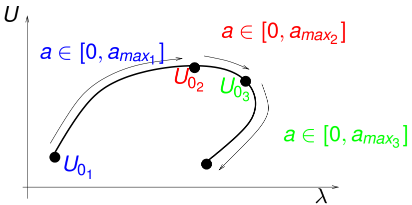
Series computation : solve a succession of linear systems that share the same stiffness matrix $\frac{\partial \mathbf{R}}{\partial \mathbf{U}}\mathbf{U}_p = F_p^{nl}(\mathbf{U}_1, \ldots, \mathbf{U}_{p-1})$



The Domain of utility of the series is the interval $[0, a_{max}]$ for which $||\mathbf{R}(\mathbf{U}(a))|| < \varepsilon_R$

A good approximation : $\mathbf{R}(\mathbf{U}_0 + \cdots + a^N \mathbf{U}_N) \simeq \mathbf{R}(\mathbf{U}_0) + a^{N+1}\mathbf{R}^{N+1}$,

So, if one requires $||a^{N+1}\mathbf{R}^{N+1}|| < \varepsilon_R$ then $a_{max} = \left(\frac{\varepsilon_R}{||\mathbf{R}^{N+1}||}\right)^{\frac{1}{N+1}}$



The complete solution branch is obtained as a succession of local Taylor series

$$\mathbf{U}(a) = \mathbf{U}_0 + a\mathbf{U}_1 + a^2\mathbf{U}_2 + \cdots + a^N\mathbf{U}_N \quad \text{avec} \quad a \in [0, a_{max}]$$

.

- piece-wise continuous representation
- auto-adaptative step length $\rightarrow$ Robustness
- no algorithmic parameter

The cornerstone

To compute the Taylor series of the solution-branch :

- Insert the Taylor series

$$u(a) = u_0 + a\, u_1 + a^2 u_2 \; + \cdots + a^N \, u_N$$

- Into the algebraic equation

$$R(u, \lambda) := u + u^2 + \frac{tan(u)}{1+u} - \lambda = 0$$

- And collect terms with the same powers !

Two techniques :

- use Automatic Differentiation to do the job : nice for the user but poor efficiency.
- do a quadratic recast of the equations : then the job becomes easy and efficient

Quadratic recast

How to recast the algebraic system $\mathbf{R}(\mathbf{U}) = 0$ in a quadratic way ?

- Goal : find auxiliary variables $\mathbf{U}_\mathrm{a}$ and $\mathbf{R}_\mathrm{f}$ such that

$$\mathbf{R}_\mathrm{f}(\mathbf{U}_\mathrm{f}) = \mathbf{C} + \mathbf{L}(\mathbf{U}_\mathrm{f}) + \mathbf{Q}(\mathbf{U}_\mathrm{f}, \mathbf{U}_\mathrm{f})$$

- with $\mathbf{U}_\mathrm{f} = (\mathbf{U}, \mathbf{U}_\mathrm{a})$,
- $\mathbf{C}$ constant,
- $\mathbf{L}$ linear,
- $\mathbf{Q}$ quadratic.

Note that $\mathbf{R}_\mathrm{f}(\mathbf{U}_\mathrm{f}) = \begin{bmatrix} \mathbf{R}(\mathbf{U}_\mathrm{f}) \\ \mathbf{R}_\mathrm{a}(\mathbf{U}_\mathrm{f}) \end{bmatrix}$ with $\frac{\partial \mathbf{R}_\mathrm{a}}{\partial \mathbf{U}_\mathrm{a}}$ invertible and $\mathbf{R}(\mathbf{U}_\mathrm{f}) = \mathbf{R}(\mathbf{U})$.

The examples of the pendulum

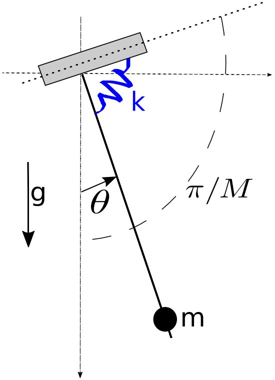
Dimensionless parameters : $m = 1$ and $g = 1$.

Energy of the system :
$H(\theta) = 1 - \cos(\theta) + \frac{k}{2}(\theta - \frac{\pi}{M})^2$

Equation of the motion :
$\ddot{\theta} + \sin(\theta) + k(\theta - \frac{\pi}{M}) = 0$

Steps :

- $M = 2$ is constant, development about $\theta = 0$:
  $\ddot{\theta} + \theta - \frac{\theta^3}{6} + k(\theta - \frac{\pi}{2}) = 0$
- $k = 0.1$ is constant, development about $\theta = 0$:
  $\ddot{\theta} + \theta - \frac{\theta^3}{6} + 0.1(\theta - \frac{\pi}{M}) = 0$
- Without simplification :
  $\ddot{\theta} + \sin(\theta) + k(\theta - \frac{\pi}{M}) = 0$



How to recast polynomials quadratically ?

Let $\mathbf{R}(\mathbf{U}) = \mathbf{R}(u, \lambda) = u^3 + u + 1 - \lambda$.

Let $v = u^2$.

$\mathbf{U} = (u, \lambda)$ and $\mathbf{U}_\mathrm{a} = v$ then $\mathbf{U}_\mathrm{f} = (\mathbf{U}, \mathbf{U}_\mathrm{a}) = (u, \lambda, v)$.

Then $\mathbf{R}_\mathrm{f}$ is defined :

$$\mathbf{R}_\mathrm{f}(\mathbf{U}_\mathrm{f}) = \mathbf{R}_\mathrm{f}(u, \lambda, v) = \begin{bmatrix} uv + u + 1 - \lambda \\ v - u^2 \end{bmatrix}$$

And the operators are $\mathbf{C} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{L}(\mathbf{U}_\mathrm{f}) = \begin{bmatrix} u - \lambda \\ v \end{bmatrix}$, $\mathbf{Q}(\mathbf{U}_\mathrm{f}, \mathbf{U}_\mathrm{f}) = \begin{bmatrix} uv \\ -u^2 \end{bmatrix}$.

Basic example : Simplified Pendulum

Pendulum subject to an angular spring at position $\theta = \frac{\pi}{2}$, developed around $\theta = 0$.
The equilibrium is given by :

$$r(\theta, k) = (\theta - \frac{\theta^3}{6}) + k(\theta - \frac{\pi}{2})$$

Definition of the auxiliary variables

$$\psi = \theta^2$$

Yields the quadratic recast

$$\mathbf{R}_{\mathrm{f}}(\mathbf{U}_{\mathrm{f}}) = \mathbf{R}_{\mathrm{f}}(\theta, k, \psi) = \begin{bmatrix} \theta - \frac{\theta\psi}{6} + k\theta - k\frac{\pi}{2}) \\ \psi - \theta^2 \end{bmatrix}$$

And the operators are $\mathbf{C} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\mathbf{L}(\mathbf{U}_{\mathrm{f}}) = \begin{bmatrix} \theta - k\frac{\pi}{2} \\ \psi \end{bmatrix}$, $\mathbf{Q}(\mathbf{U}_{\mathrm{f}}, \mathbf{U}_{\mathrm{f}}) = \begin{bmatrix} -\frac{\theta\psi}{6} \\ -\theta^2 \end{bmatrix}$.

How to recast fractions quadratically ?

Let $\mathbf{R}(\mathbf{U}) = \mathbf{R}(u, \lambda) = \frac{1}{u} + u - \lambda$.

Let $v = \frac{1}{u}$. v can be defined implicitly through $uv = 1$.

Then $\mathbf{R}_\mathrm{f}$ is defined :

$$\mathbf{R}_\mathrm{f}(\mathbf{U}_\mathrm{f}) = \mathbf{R}_\mathrm{f}(u, \lambda, v) = \begin{bmatrix} v + u - \lambda \\ uv - 1 \end{bmatrix}$$

And the operators are $\mathbf{C} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, $\mathbf{L}(\mathbf{U}_\mathrm{f}) = \begin{bmatrix} u + v - \lambda \\ 0 \end{bmatrix}$, $\mathbf{Q}(\mathbf{U}_\mathrm{f}, \mathbf{U}_\mathrm{f}) = \begin{bmatrix} 0 \\ uv \end{bmatrix}$.

Here, $\mathbf{R}_\mathrm{a}(\mathbf{U}_\mathrm{f}) = uv - 1$ and $\frac{\partial \mathbf{R}_\mathrm{a}}{\partial \mathbf{U}_\mathrm{a}} = \frac{\partial \mathbf{R}_\mathrm{a}}{\partial v} = u$. It is invertible if and only if $u \neq 0$.

Simple example : Another Simplified Pendulum

Pendulum subject to an angular spring at position $\theta = \frac{\pi}{M}$, developed around $\theta = 0$.
The equilibrium is given by :

$$r(\theta, M) = (\theta - \frac{\theta^3}{6}) + 0.1(\theta - \frac{\pi}{M})$$

Definition of the auxiliary variables

$$\begin{aligned} \psi &= \theta^2 \\ M_{inv} &= \frac{1}{M} \end{aligned}$$

Yields the quadratic recast

$$\mathbf{R}_{\mathrm{f}}(\mathbf{U}_{\mathrm{f}}) = \mathbf{R}_{\mathrm{f}}(\theta, M, \psi, M_{inv}) = \begin{bmatrix} \theta - \frac{\theta\psi}{6} + 0.1(\theta - M_{inv}\pi) \\ \psi - \theta^2 \\ M_{inv}M - 1 \end{bmatrix}$$

How to recast **everything else** quadratically ?

Let $\mathbf{R}(\mathbf{U}) = \mathbf{R}(u,\lambda) = u - \tan(u) - \lambda$. Let $t = \tan(u)$ and let $z = 1 + t^2$. t and z can be defined by the system :

$$\begin{aligned} \mathrm{d}t &= z\,\mathrm{d}u \\ z &= 1 + t^2 \end{aligned}$$

These equations are quadratic with respect to the Taylor coefficients of u,t and z.
Then $\mathbf{R}_\mathrm{f}$ and its differential form $\mathrm{d}\mathbf{R}_\mathrm{f}$ are defined :

$$\mathbf{R}_\mathrm{f}(\mathbf{U}_\mathrm{f}) = \mathbf{R}_\mathrm{f}(u,\lambda,t,z) = \begin{bmatrix} u - t - \lambda \\ t - \tan(u) \\ z - 1 - t^2 \end{bmatrix} \quad \mathrm{d}\mathbf{R}_\mathrm{f}(\mathbf{U}_\mathrm{f},\mathrm{d}\mathbf{U}_\mathrm{f}) = \begin{bmatrix} \text{Not needed} \\ \mathrm{d}t - z\,\mathrm{d}u \\ \text{Not needed} \end{bmatrix}$$

And the operators are $\mathbf{C} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{L}(\mathbf{U}_\mathrm{f}) = \begin{bmatrix} u - t - \lambda \\ 0 \\ z \end{bmatrix}$, $\mathbf{Q}(\mathbf{U}_\mathrm{f},\mathbf{U}_\mathrm{f}) = \begin{bmatrix} 0 \\ 0 \\ -t^2 \end{bmatrix}$,

$\mathrm{d}\mathbf{L}(\mathrm{d}\mathbf{U}_\mathrm{f}) = \begin{bmatrix} 0 \\ \mathrm{d}t \\ 0 \end{bmatrix}$ and $\mathrm{d}\mathbf{Q}(\mathbf{U}_\mathrm{f},\mathrm{d}\mathbf{U}_\mathrm{f}) = \begin{bmatrix} 0 \\ -z\,\mathrm{d}u \\ 0 \end{bmatrix}$.

Transcendental example : Pendulum

Pendulum subject to an angular spring at position $\theta = \frac{\pi}{M}$.
The equilibrium is given by :

$$r(\theta, k, M) = \sin(\theta) + k(\theta - \frac{\pi}{M})$$

Definition of the auxiliary variables, together with the differentiated forms (when needed) :

$$\begin{aligned} s &= \sin(\theta) & \mathrm{d}s &= c\mathrm{d}\theta \\ c &= \cos(\theta) & \mathrm{d}c &= -s\mathrm{d}\theta \\ M_{inv} &= \frac{1}{M} & & \end{aligned}$$

Define $\mathbf{U}_\mathrm{f} = (\theta, k, M, s, c, M_{inv})$ yields the "quadratic" recast

$$\mathbf{R}_\mathrm{f}(\mathbf{U}_\mathrm{f}) = \begin{bmatrix} s + k(\theta - M_{inv}\pi) \\ s - \sin(\theta) \\ c - \cos(\theta) \\ M_{inv}M - 1 \end{bmatrix} \quad \mathrm{d}\mathbf{R}_\mathrm{f}(\mathbf{U}_\mathrm{f}, \mathrm{d}\mathbf{U}_\mathrm{f}) = \begin{bmatrix} \text{Not needed} \\ \mathrm{d}s - c\mathrm{d}\theta \\ \mathrm{d}c + s\mathrm{d}\theta \\ \text{Not needed} \end{bmatrix}$$

End of the Algebraic elements of theory.

Taylor series algebra

- Product of Taylor series $u(a) \times v(a)$:

$$\left(u_0 + au_1 + a^2u_2 + \cdots + a^N u_N\right) \times \left(v_0 + av_1 + a^2v_2 + \cdots + a^N v_N\right)$$
$$= u_0 v_0 + a(u_1 v_0 + u_0 v_1) + a^2(u_2 v_0 + u_1 v_1 + u_0 v_2) + \cdots + a^N \sum_{j=0}^{N} u_{N-j} v_j$$

It is truncated at order N.

- Differentiation of Taylor series $\frac{\partial u}{\partial a}(a)$:

$$\frac{\partial}{\partial a}\left(u_0 + au_1 + a^2u_2 + \cdots + a^N u_N\right) = u_1 + 2au_2 + 3a^2u_3 + \cdots + Na^{N-1}u_N$$

The constant coefficient u_0 is not anymore in the development, that goes now up to order $N-1$.

Manlab 4.0

Let $R(\mathbf{U}) = 0$ be

$$\begin{aligned} r_1(u_1, u_2, \lambda) = \quad & 2u_1 - u_2 + 100\frac{u_1}{1+u_1+u_1^2} - \lambda & = 0 \\ r_2(u_1, u_2, \lambda) = \quad & 2u_2 - u_1 + 100\frac{u_2}{1+u_2+u_2^2} - (\lambda + \mu) & = 0 \end{aligned}$$

Introduce the auxiliary variables :
$$\begin{aligned} v_1 &= 1 + u_1 + u_1 u_1 \\ v_2 &= 1 + u_2 + u_2 u_2 \\ v_3 &= 1/v_1 \\ v_4 &= 1/v_2 \end{aligned}$$

- All these expression are quadratic, or easily made quadratic
- "linear declaration rule" : an auxiliary variable v_i cannot appear on the left hand side before it has been explicitely defined as $v_i = f(\mathbf{U}, v_1, v_2, \ldots, v_{i-1})$. Ensures that $\frac{\partial \mathbf{R}_a}{\partial \mathbf{U}_a}$ is invertible.

Let $\mathbf{U}_a = [v_1, v_2, v_3, v_4]$ be the vector of auxiliary variables
Let $\mathbf{U}_f = [\mathbf{U}, \mathbf{U}_a]$

Manlab 4.0

The original system $R(\mathbf{U}) = 0$ is replaced by the equivalent quadratic one $\mathbf{R}(\mathbf{U}_f)$

$$
\begin{array}{lll}
r_1 := & 2u_1 - u_2 + 100u_1 v_3 - \lambda & = 0 \\
r_2 := & 2u_2 - u_1 + 100u_2 v_3 - (\lambda + \mu) & = 0 \\
r_{aux1} := & v_1 - 1 + u_1 + u_1 * u_1 & = 0 \\
r_{aux2} := & v_2 - 1 + u_2 + u_2 * u_2 & = 0 \\
r_{aux3} := & v_3 * v_1 - 1 & = 0 \\
r_{aux4} := & v_4 * v_2 - 1 & = 0
\end{array}
$$

Tensor formalism : this quadratic system may be written

$$R_i = C_i + L_{ij} U_j + Q_{ijk} U_j U_k \qquad i, j, k = 1, 2, \dots n$$

with C,L,Q being tensors of order 1, 2 and 3
Here, we have 7 components C_i, 49 components L_{ij} and 343 components Q_{ijk}.

But most of them are zero !

Manlab 4.0 : Sparse tensor formalism

The sparse tensor C, L and Q are defined by the following lists (as in Matlab for a sparse matrix)

- order 1 tensor C

```
iC= [ 2   5   6 ]
vC= [-μ -1   -1 ]
```

- order 2 tensor L

```
iL= [1   1   1   2   2   2   3   3   4   4   5   6 ]
jL= [1   2   7   1   2   7   1   3   2   4   5   6 ]
vL= [2 -1  -1 -1   2   -1 -1   1 -1   1   1   1 ]
```

- order 3 tensor Q

```
iQ= [ 1   2   3   4   5   6 ]
jQ= [ 1   2   1   2   3   4 ]
kQ= [ 5   6   1   2   5   6 ]
vQ= [100 100 -1 -1   1   1 ]
```

In Manlab 4.0, these lists are automatically generated from the quadratic system.

Manlab 4.0 : Sparse tensor formalism

How to get the lists defining the sparse tensor, from the quadratic expression $R(X) := 0$?

Polarization formula :

$$C = R(0)$$

$$L(X) = \frac{1}{2}\left(R(X) - R(-X)\right)$$

$$Q(X, Y) = \frac{1}{4}\left(R(X+Y) - R(X-Y) - R(Y) + R(-Y)\right)$$

Manlab 4.0 : Sparse tensor formalism

Using these lists, the computation of the residual vector $R(\mathbf{U}) = C + L(\mathbf{U}) + Q(\mathbf{U}, \mathbf{U})$ stand in one (Matlab) line.

```
R =sparse(iC,ones(1,size(iC,2)),vC',neq,1)
+ sparse(iL,ones(1,size(iL,2)),vL'.*U(jL),neq,1)
+ sparse(iQ,ones(1,size(iQ,2)),vQ'.*(U(jQ).*U(kQ)),neq,1)
```

For the jacobian matrix $dRdU = L(.) + Q(\mathbf{U}, .) + Q(., \mathbf{U})$

```
dRdU = sparse(iL,jL,sys.vL,neq,ninc)
+ sparse(iQ,kQ,vQ'.*U(jQ),neq,ninc)
+ sparse(iQ,jQ,vQ'.*U(kQ),neq,ninc)
```

Manlab 4.0 : Condensation

The linear problem to be solved at each order p reads :

$$\begin{bmatrix} B & A_{aux} \\ A & C \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{U}_{aux} \end{bmatrix} = \begin{bmatrix} F_{aux\ p} \\ F_p \end{bmatrix} \qquad \begin{matrix} \leftarrow R_{aux} \\ \leftarrow R \end{matrix}$$

Thanks to the "linear declaration rule", the matrix A_{aux} is triangular which allows an easy and cheap block solving

We first solve

$$\left[A - C\ A_{aux}^{-1}\ B\right]\left[U\right] = \left[F_p - C\ A_{aux}^{-1} F_{aux\ p}\right]$$

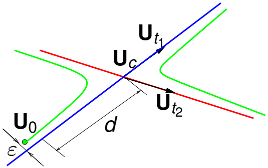
where $\left[A - C\ A_{aux}^{-1}\ B\right]$ is the jacobian matrix of the original (non quadratic) system.

Bifurcation detection using series analysis

Numerical evidence : near a simple bifurcation, a geometric series emerge in the Taylor series.

$$\mathbf{U}(a) = \mathbf{U}_0 + a\,\mathbf{U}_1 + a^2\,\mathbf{U}_2 + a^3\,\mathbf{U}_3 + \dots$$

First order analysis : expression of a perturbed branches near a simple bifurcation [Cochelin & Médale, 2013]



$$\mathbf{U}(a) = \mathbf{U}_0 + \; a\,\mathbf{U}_{t_1} - \varepsilon\,\frac{\frac{a}{d}}{(1-\frac{a}{d})}\mathbf{U}_{t_2}$$

- $\frac{\frac{a}{d}}{1-\frac{a}{d}} = \frac{a}{d} + (\frac{a}{d})^2 + (\frac{a}{d})^3 + \cdots$, a geometric serie with common ratio $\frac{1}{d}$
- After each Taylor series computation, we look for an emerging geometric series. When detected , it is extracted, completed to infinity and replaced by a fraction

$$\mathbf{U}(a) = \underbrace{\mathbf{U}_0 + a\,\hat{\mathbf{U}}_1 + a^2\,\hat{\mathbf{U}}_2 + \cdots + a^{n-1}\,\hat{U}_{n-1}}_{\hat{\mathbf{U}}(a)\ \ \text{cleaned series}} + \frac{a}{d}\left(\frac{1}{1-\frac{a}{d}}\right)\mathbf{U}_{scale}$$

We get d, $\mathbf{U}_{t2}$ and can go further the bifurcation thanks to the cleaned series.